

Expected Consumption Growth, Stochastic Volatility and Long-term Real Interest Rates

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Abstract

This paper estimates a joint econometric model of consumption growth and long-term real interest rates with stochastic volatility based on data from the U.K. The model imposes no-arbitrage condition on the term structure of real interest rates and extends the standard long-run risk model which assumes constant market prices of risk. We find that both the long-run consumption risk and the volatility risk are priced in long-term real bond yields. The long-run consumption risk dominates the volatility risk and drives most of the movements of bond risk premiums. In contrast to the standard long-run risk model, we find that a counter-cyclical time-varying market price of risk, not the stochastic volatility, is the primary source of time-variations in bond risk premiums, accounting for more than 70% of the variance of the risk premium on the 10-year real bond.

JEL Classification: G12, E43

Key Words: consumption, long-run risk, real interest rates

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1 Introduction

Economic theories suggest a close relation between real interest rates and consumption through the elasticity of intertemporal substitution. The expected consumption growth is a key state variable determining real interest rates.¹ Even though the standard consumption-based asset pricing model is often rejected by financial market data, recent studies that feature more general specifications of investor's preferences such as those with recursive utility or habit persistence, however, have documented more supporting evidence for consumption risks to explain major financial market phenomena. Among studies that focus on the term structure of interest rates, both Wachter (2006) and Buraschi and Jiltsov (2007) find that a consumption-based model with the habit utility of Campbell and Cochrane (1999) can account for many features of the nominal term structure of interest rates. Piazzesi and Schneider (2006) uses a representative agent asset pricing model with recursive utility preferences of Epstein and Zin (1989) and Weil (1989) to examine the roles of inflation and long-run consumption growth in determining risk premiums on the U.S. Treasury bonds. Using the same recursive utility preferences, Gallmeyer et al. (2007) derives the equilibrium yield curve that conforms with the standard affine term structure model and is able to relate factor loadings and market prices of risk to deep structural parameters.

The state variables in these models include not only consumption growth, but also a latent variable that represents either an exogenous consumption habit or the expected long-run consumption growth. In particular, in the long-run risk model of Bansal and Yaron (2004) and Bansal (2007), consumption growth includes a small but persistent long-run component. Fluctuations of this long-run component, together with stochastic volatilities of consumption growth, drive financial markets as recursive utility preferences generate heightened concerns about long-run growth prospects of the

¹Some early empirical studies that investigate the relation between real interest rates and consumption growth include Hall (1988), Harvey (1988) and Chapman (1997) among many others.

economy and the time-varying levels of economic uncertainty. A recent application of the long-run risk model is Bansal and Shaliastovich (2013) which shows that the model goes a long way to account for many stylized facts in bond and foreign exchange markets.

Motivated by these results, the current paper estimates a joint econometric model of consumption growth and long-term real interest rates based on data from the U.K. The model imposes no-arbitrage condition on the term structure of real interest rates and hence nests the standard long-run risk model. We seek to find direct empirical evidence of the aggregate consumption growth as a major risk factor driving the real bond market and to understand more precisely the roles of expected consumption growth and growth volatility in determining the long-end of the real yield curve. Long-term interest rates encode information about investor's intertemporal marginal rate of substitution. And focusing on real interest rates allows us to concentrate on consumption growth while abstracting from the effect of inflation.² Unlike the tightly specified equilibrium asset pricing models, this paper doesn't attempt to identify and estimate the structural parameters that characterize investor's preferences. Using the no-arbitrage condition, the model adopts a more flexible specification of market prices of risk. It lies within the broad class of dynamic affine models of the term structure of interest rates.³ Retaining such econometric flexibilities is important if we are to decode information from asset prices about long-run economic growth prospects as well as the required risk compensations.

The paper contributes to a growing literature on the estimation of long-run risk models, including Bansal, Gallant and Tauchen (2007), Chen, Favilukis and Ludvigson (2013) and Schorfheide et al. (2013) among others. These studies focus on stock market returns instead of the term structure

²As shown in Gallmeyer et al. (2007), the empirical properties of equilibrium models of the nominal term structure of interest rates depends critically on the econometric assumptions about inflation. In the case of stock market returns, additional assumptions about the dividend process are needed.

³See Duffie and Kan (1996) and Dai and Singleton (2000) for the standard affine models. Their extensions can be found in Duffie (2002), Duarte (2003) and Cheridito, Filipovic and Kimmel (2007) among others.

of interest rates. Doh (2013) uses Bayesian methods to estimate a Gaussian model of the nominal term structure under the long-run risk while assuming a constant growth volatility. Bansal, Kiku and Yaron (2012) and Beeler and Campbell (2012) provide calibration-based evaluations of the long-run risk model. In all these studies, the stochastic discount factor for asset pricing is derived from Epstein-Zin recursive utility preferences. The log-linearized Euler equation leads to constant market prices of risk. Risk premiums are time-varying solely because of stochastic volatilities of consumption growth. As pointed out by Beeler and Campbell (2012), while the model is able to account for many stylized facts of asset returns, the model also implies a downward-sloping real yield curve. A quick inspection of the data from inflation-indexed bond yield from the U.K., however, indicates that the real yield curve is in fact upward-sloping for most parts of the sample period between 1985 and 2011 (see Figure 6 below). The average yield spread between a 15-year real bond and a 5-year real bond is about 30 basis points. Moreover, the yield spread varies greatly over time, suggesting possible time-variations in bond risk premiums.⁴ Using a flexible specification of the market price of risk, our model is able to match the upward-sloping yield curve. More importantly, we find that the time-varying market price of risk, not the stochastic volatility, is the primary source of time-variations in bond risk premiums. In contrast, a standard long-run risk model would attribute all the time-variations in bond risk premiums to changes levels of growth uncertainty.

The rest of the paper is organized as follows. Section 2 provides summary statistics about consumption growth and long-term real interest rates. Section 3 presents the arbitrage-free dynamic model of the real yield curve under the long-run consumption risk. The estimation and empirical results are discussed in section 4. Section 5 contains some concluding remarks.

⁴A another factor that determines the yield spread is the expected future short rate.

2 Data and Summary Statistics

We study the joint dynamics of aggregate consumption growth and long-term real interest rates. Ex-ante real interest rates are not observable. The best approximations are yields on inflation index-linked government bonds. The U.K. market for inflation index-linked government debts was started in 1981 and has the longest time series on such yields. Yields on Treasury Inflation Protected Securities (TIPS) in the U.S. also provide close approximations to real interest rates. But the U.S. market was started in the 1990s and has much shorter time series. Since the data on aggregate consumption are available on a quarterly basis, we use the U.K. data on real interest rates and consumption in this paper.⁵ Consumption data are obtained from International Financial Statistics (IFS). Consumption growth rates are calculated as quarterly percentage changes of seasonally adjusted per-capita real consumption on non-durable goods and services. Data on real interest rates are obtained from Bank of England which also provides detailed explanations of the estimation of zero-coupon yields from bond prices. We only use the long end of the real yield curve (5-year to 15-year) in this study. One reason is that the U.K. index-linked bonds have their coupon and principal payments effectively linked to the Retail Price Index published approximately eight months prior to the payment date. While this “indexation lag problem” may create serious errors in the estimates of short-term real interest rates, its effect on the long end of the yield curve should be negligible. We use data from the first quarter of 1985 to the end of 2011. Interest rates are collected at the beginning of each quarter. We exclude the data from 1981 to 1984 mainly because the market for index-linked bonds was known to be not very liquid in the early years of its development, and the bond yields might include a significant liquidity premium.

In Table 1 we report the summary statistics on consumption growth and the long-term real interest rates. Between 1985 and 2011, aggregate con-

⁵Other studies of the real term structure that also utilize the U.K. index-linked bond yields include Evans (1998, 2003) and Seppälä (2005) among others.

consumption grows at an annual rate of about 1.87%. Compared to long-term real interest rates, consumption growth is much more volatile with a standard deviation of 3.2%. The standard deviations of long-term interest rates are all smaller than 1%. During the sample period, consumption growth is positively correlated with an autocorrelation coefficient of 0.1589. In contrast, interest rates are very persistent. The autocorrelation coefficients for the long-term real interest rates range from 0.78 to 0.89. The real yield curve has a positive slope on average during 1985-2011. The mean of the 5-year rate is 3.08%, while the mean of the 15-year rate is 3.38%. This fact posts a challenge for the standard long-run risk model which implies a downward-sloping real yield curve.⁶

One often observed feature of the term structure of interest rates is that interest rate volatility doesn't seem to attenuate as maturity increases. This is also true about the long-term real interest rates. As we can see from Table 1, as interest rate maturity increases from 5 years to 15 years, the standard deviation only declines slightly from 0.65% to 0.60%. Our term structure model is able to capture these empirical properties of the real yield curve.

The close relation between real interest rates and consumption growth can be seen from Table 2, where we report the cross correlations between the long-term real interest rates and the consumption growth rate. The table clearly shows that, consistent with economic theories, real interest rates are strongly and positively correlated with consumption growth. In particular, the real interest rates in our sample are most correlated with 1-year or 2-year ahead consumption growth (Δc_{t+4} or Δc_{t+8}). For example, the correlation between the 5-year real interest rate, $R_{5,t}$, with Δc_{t+8} is 0.34. The correlation between the real interest rates and lagged consumption growth is much weaker. The highest correlation is between $R_{5,t}$ and Δc_{t-4} at

⁶Some earlier studies such as Piazzesi and Schneider (2006) have reported a downward-sloping real yield curve for the U.K. indexed bonds between 1995 and 2006 based on monthly data. We also find that the yield curve is downward-sloping for the same sample period using quarterly data. But for the whole sample period and for most of the sub-sample periods, the yield curve slope is positive. See Figure 6 and discussions in Section 4.

about 0.23. These results suggest that, if consumption growth is predictable, the state variables that predict future consumption growth are likely to play an important role in determining real interest rates as well.

To see that long-term real interest rates indeed predict consumption growth, we regress future consumption growth on lagged 5-year real interest rate and lagged consumption growth. The results are reported in Table 3. We can see that the real interest rate has significant forecasting power for consumption growth in the next quarter, $\Delta c_{t+1} = \log c_{t+1} - \log c_t$, while the lagged consumption growth does not. We also regress long-run consumption growth on the lagged interest rate and consumption growth. A 4-quarter or 8-quarter moving average of the quarterly growth rate is used to measure long-run consumption growth. We find that both the 5-year rate and lagged consumption growth have significant forecasting powers for future long-run consumption growth. The R-square from the forecasting regression is around 20% and 37% respectively, and a higher level of real interest rates predicts higher future consumption growth. Early empirical studies on the predictive power of the yield curve for future economic activities have focused almost exclusively on the slope of the yield curve. Our result is consistent with that of Ang and Piazzesi (2006) which shows that the level of interest rates may have more predicative power for GDP growth than any yield spread does.

3 A Consumption-Based Term Structure Model

Motivated by the empirical results above, in this section, we construct a simple consumption-based asset pricing model of the terms structure of real interest rates. The model allows us to study the joint dynamics of consumption growth and the long-end of the real yield curve while respecting the no-arbitrage condition on the cross-section of the real interest rates.

3.1 State variables

We assume that the relevant state variables for the pricing of long-term real bonds are the expected consumption growth and the stochastic consumption growth volatility. In particular, we postulate that

$$\Delta c_{t+1} = \mu_0 + \mu_t + \sqrt{z_t} \varepsilon_{t+1} \quad (1)$$

where μ_t represents a small and potentially very persistent predictable component of (demeaned) consumption growth, the conditional variance of consumption growth is driven by a *positive* and also potentially very persistent state variable z_t , and ε_{t+1} is an i.i.d. $\mathcal{N}(0,1)$ shock to consumption growth. In this paper we are primarily interested in the impact of the expected consumption growth, μ_t , and the growth volatility, z_t , on the long-end of the real yield curve.

We assume

$$\mu_{t+1} = \phi \mu_t + \sigma_\mu \nu_{t+1} \quad (2)$$

where $|\phi| < 1$, $\sigma_\mu > 0$, ν_{t+1} is also i.i.d. $\mathcal{N}(0,1)$ shock to the expected consumption growth. Following the standard long-run risk model such as that in Bansal and Yaron (2004) and Bansal, Kiku and Yaron, (2012), we assume ε_{t+1} and ν_{t+1} are mutually independent. One key difference is that we assume that μ_{t+1} has constant volatility for simplicity. Allowing the state variable z_t to also drive the volatility of μ_{t+1} as in Bansal and Yaron (2004) and Bansal, Kiku and Yaron (2012) will change the factor loadings in a term structure model. This would be useful if we are interested in estimating deep structural parameters of an utility function. Since we will estimate a reduced form model based on no-arbitrage condition, the homoscedasticity assumption has little impact on the empirical performance of our model.⁷ A more interesting case is that in Drechsler and Yaron (2011), Schorfheide, Song and Yaron (2013) and others where a separate state variable drives

⁷Since the other state variable z_{t+1} has stochastic volatility as explained below, so is our term structure model.

the stochastic volatility of the *expected* consumption growth. This will introduce a third factor to the term structure model. However, as we will see below, a model with just two factors (the expected consumption growth and consumption growth volatility) is able to explain almost all of the dynamics of the long-end of the real yield curve.⁸

The state variable, z_t , drives the stochastic volatility of consumption growth. Since it needs to be positive, we assume that z_t follows an autoregressive gamma process as in Gourieroux and Jasiak (2006) and Dai, Le and Singleton (2010).⁹ That is, conditional on z_t ,

$$\frac{z_{t+1}}{\psi} \sim \text{gamma}(\delta + \mathbf{P}) \quad (3)$$

where

$$\mathbf{P} \sim \text{Poisson} \left(\frac{\rho z_t}{\psi} \right)$$

for some positive parameters $\delta > 0$, $0 < \rho < 1$ and $\psi > 0$. We also assume that z_{t+1} is conditionally independent of μ_{t+1} and Δc_{t+1} . To prevent z_t from attaining the zero lower bound, we further restrict $\delta > 1$ in the empirical estimation below.¹⁰ Note that the conditional mean and variance of z_{t+1} are given by

$$\mathbb{E}_t(z_{t+1}) = \psi\delta + \rho z_t \quad (4)$$

$$\text{Var}_t(z_{t+1}) = \psi^2\delta + 2\psi\rho z_t \quad (5)$$

As a result, our dynamic term structure model of real interest rates also

⁸Schorfheide, Song and Yaron (2013) shows that the volatility of the *expected* consumption growth accounts for very little of the variations in the real short rate.

⁹The conditional density of z_{t+1} given z_t takes the following form $f(z_{t+1}|z_t) = \exp(-\frac{z_{t+1}}{c}) \sum_{k=0}^{\infty} \left[\frac{1}{c} \left(\frac{z_{t+1}}{c}\right)^{\delta+k-1} \frac{1}{\Gamma(\delta+k)} \frac{\exp(-\rho z_t/c) (\rho z_t/c)^k}{k!} \right]$. In continuous-time limit, z_t converges to the square-root process $dz(t) = k(\theta - z(t))dt + \sigma\sqrt{z(t)}dB(t)$ where $k\Delta t = 1 - \rho$, $\frac{1}{2}\sigma^2\Delta t = c$ and $\frac{2k\theta}{\sigma^2} = \delta$. See Gourieroux and Jasiak (2006) for more detailed discussions of the properties of z_t . Dai, Le and Singleton (2010) provides a multivariate extension of the autoregressive gamma process.

¹⁰See Cheridito, Filipovic and Kimmel (2007) and Dai, Le and Singleton (2010) for more discussions on this issue.

features stochastic volatility.

3.2 The stochastic discount factor

Instead of making explicit assumptions about investor's intertemporal utility function as in equilibrium asset pricing models, we derive the term structure of interest rates using the no-arbitrage approach in this paper.¹¹ The less stringent condition of no-arbitrage implies that there exists a positive stochastic discount factor, $M_{t,t+1}$, such that for an asset with a payoff D_{t+1} at $t + 1$, its time- t price is given by

$$P_t = E_t(M_{t,t+1}D_{t+1}) \quad (6)$$

Let $X_t = (z_t, \mu_t)'$. Following Dai, Le and Singleton (2010), we postulate a general specification for the stochastic discount factor as follows,

$$M_{t,t+1} = e^{-r_t} \frac{e^{-\lambda_t' X_{t+1}}}{L_t^P(\lambda_t)} \quad (7)$$

where r_t is the one-period risk-free rate, $\lambda_t = (\lambda_{z,t}, \lambda_{\mu,t})'$, a 2×1 vector of market prices of risk, and $L_t^P(\lambda_t)$ is the conditional (two-sided) Laplace transforms of X_{t+1} . Because of the conditional independence between z_{t+1} and μ_{t+1} , $L_t^P(\lambda_t)$ can be obtained as, for any $k = (k_z, k_\mu)'$,

$$L_t^P(k) = E_t \left(e^{-k' X_{t+1}} \right) = e^{-a(k_z) - b(k_z) z_t} \times e^{-k_\mu \bar{\mu}_t^P + \frac{1}{2} k_\mu^2 \sigma_\mu^2} \quad (8)$$

where $\bar{\mu}_t^P = \phi \mu_t$ is the conditional mean of μ_{t+1} under the physical probability measure \mathbb{P} , and $a(k_z)$ and $b(k_z)$ in (8) are two nonlinear functions of k_z given by

$$a(k_z) = \delta \log(1 + k_z \psi), \quad b(k_z) = \frac{k_z \rho}{1 + k_z \psi}$$

The appendix shows that the specification of the stochastic discount factor

¹¹The appendix explains the connection between our model and an equilibrium model with Epstein-Zin recursive preferences.

above is closely related to the inter-temporal marginal rate of substitution (IMRS) in the long-run risk models such as Bansal and Yaron (2004) and Bansal, Kiku and Yaron (2012). The key difference, however, is that while the market prices of risk are constant in the long-run risk models, they can be flexible functions of the state variables in our model.

3.3 Risk neutral probability measure and market prices of risk

The solution to the real yield curve can be obtained by a change of probability measure. We define the risk-neutral probability measure by the following Radon-Nykody derivative

$$\xi_{t,t+1} = \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right)_{t,t+1} = \frac{e^{-\lambda'_t X_{t+1}}}{L_t^P(\lambda_t)} \quad (9)$$

It then follows that the conditional Laplace transforms of X_{t+1} under the risk-neutral probability measure \mathbb{Q} is given by

$$L_t^Q(k) = \frac{L_t^P(k + \lambda_t)}{L_t^P(\lambda_t)} = e^{-a_t^*(k_z) - b_t^*(k_z)z_t} \times e^{-k_\mu \bar{\mu}_t^Q + \frac{1}{2}k_\mu^2 \sigma_\mu^2} \quad (10)$$

where

$$\psi_t^* = \frac{\psi}{1 + \lambda_{z,t} \psi}, \quad \rho_t^* = \frac{\rho}{(1 + \lambda_{z,t} \psi)^2}$$

and

$$a_t^*(k_z) = \delta \log(1 + k_z \psi_t^*) \quad b_t^*(k_z) = \frac{k_z \rho_t^*}{1 + k_z \psi_t^*}$$

$\bar{\mu}_t^Q$ is the conditional mean of μ_{t+1} under the risk-neutral probability measures \mathbb{Q} and is given by

$$\bar{\mu}_t^Q = \bar{\mu}_t^P - \sigma_\mu^2 \lambda_{\mu,t} \quad (11)$$

We now make two assumptions about the market prices of risk in order to get an analytical solution of the term structure of real interest rates.

First, we assume that $\lambda_{z,t}$ is constant,

$$\lambda_{z,t} = \lambda_z \quad (12)$$

In this case, we have

$$L_t^Q(k) = e^{-a^*(k_z) - b^*(k_z)z_t} \times e^{-k_\mu \bar{\mu}_t^Q + \frac{1}{2}k_\mu^2 \sigma_\mu^2} \quad (13)$$

where

$$a^*(k_z) = \delta \log(1 + k_z \psi^*) \quad b^*(k_z) = \frac{k_z \rho^*}{1 + k_z \psi^*} \quad (14)$$

$$\psi^* = \frac{\psi}{1 + \lambda_z \psi}, \quad \rho^* = \frac{\rho}{(1 + \lambda_z \psi)^2} \quad (15)$$

In other words, z_t still follows an autoregressive gamma process that is conditionally independent of μ_t under \mathbb{Q} .

Secondly, we assume μ_{t+1} is also AR(1) under \mathbb{Q} with its conditional mean given by

$$\bar{\mu}_t^Q = \phi_0^* + \phi_1^* \mu_t \quad (16)$$

This is equivalent to assume that

$$\sigma_\mu^2 \lambda_{\mu,t} = -\phi_0^* - (\phi_1^* - \phi_1) \mu_t \quad (17)$$

As shown in the appendix, the standard long-run risk model assumes $\lambda_{\mu,t}$ is constant and therefore restrict $\phi_1 = \phi_1^*$. In our case, however, ϕ_1 and ϕ_1^* are not necessarily the same and are determined by the interest rate and consumption data. In addition to stochastic growth volatility, this time-varying market price of risk introduces a new source of time-varying risk premiums.

3.4 An affine model of the real yield curve

Now we consider the market for zero-coupon bonds that are free of default risk. Let $P_{n,t}$ denote the real price at time t of a n -period bond that pays one unit of consumption goods when it matures. In the absence of arbitrage opportunities, we must have

$$P_{n,t} = E_t^P(M_{t,t+1}P_{n-1,t+1}) = e^{-r_t}E_t^Q(P_{n-1,t+1}) \quad (18)$$

where the first expectation $E_t^P(\cdot)$ is taken with respect to the physical probability measure \mathbb{P} and the second expectation $E_t^Q(\cdot)$ is taken with respect to the risk-neutral probability measure \mathbb{Q} .

The model is completed by assuming that the short-term interest rate, r_t , is given by

$$r_t = A_1 + B_1\mu_t + C_1z_t \quad (19)$$

and $P_{n,t}$ can be obtained as

$$P_{n,t} = e^{-A_n - B_n\mu_t - C_nz_t} \quad (20)$$

where the coefficient A_n and B_n are determined by the following system of difference equations, starting from $A_0 = B_0 = C_0 = 0$,

$$A_n = A_1 + A_{n-1} + a^*(C_{n-1}) + \phi_0^*B_{n-1} - \frac{1}{2}B_{n-1}^2\sigma_\mu^2 \quad (21)$$

$$B_n = B_1 + \phi_1^*B_{n-1} \quad (22)$$

$$C_n = C_1 + b^*(C_{n-1}) \quad (23)$$

where

$$a^*(C_{n-1}) = \delta \log(1 + \psi^*C_{n-1})$$

$$b^*(C_{n-1}) = \frac{\rho^*C_{n-1}}{1 + \psi^*C_{n-1}}$$

Continuously compounding n -period interest rate, $R_{n,t}$, is defined by

$$P_{n,t} = e^{-nR_{n,t}}$$

and we have

$$R_{n,t} = \frac{A_n}{n} + \frac{B_n \mu_t}{n} + \frac{C_n z_t}{n} \quad (24)$$

4 Estimation and Empirical Results

We estimate the joint dynamics of consumption growth and long-term real interest rates based on the term structure model developed above. As discussed in Section 2, the data on consumption are quarterly percentage changes of seasonally adjusted per-capita real consumption of non-durable goods and services in U.K. from 1985 to 2011. We use yields on 5-year, 10-year and 15-year inflation index-linked zero-coupon bonds as ex-ante real interest rates. We assume that 5-year and 10-year bonds are priced without error. We can, therefore, solve for $(z_t, \mu_t)'$ using the 5-year and 10-year rates, $(R_{5,t}, R_{10,t})'$. We assume that the 15-year bond is priced with an error that has a normal distribution $\mathcal{N}(0, \sigma_{15}^2)$.

These assumptions together with those on the distribution of state variables Δc_t , μ_t and z_t (under probability measure \mathbb{P}) in Section 3.1 enable us to write down the joint likelihood function for $\{\Delta c_t, R_{5,t}, R_{10,t}, R_{15,t}\}$ where $R_{n,t}$ denotes n -year real interest rate. The parameters to be estimated in this models include: (1) parameters that govern the \mathbb{P} -distribution of the state variables $\{\delta, \rho, \psi, \mu_0, \phi, \sigma_\mu\}$; (2) parameters that determine the market prices of risk or the \mathbb{Q} -distribution of the state variables $\{\lambda_z, \phi_0^*, \phi_1^*\}$; (3) parameters that determine the short-term interest rate $\{A_1, B_1, C_1\}$; and (4) standard deviations of the pricing errors for 15-year, $\{\sigma_{15}\}$.

Given the affine structure of the term structure model, it is well known that the model is invariant with respect to certain linear transforms and it is necessary to normalize some parameters to achieve identification. In

our case, identification can be achieved by simply setting $C_1 = 1$. We are then left with a total of 12 parameters that can be estimated by maximum likelihood method. The results are reported in Table (4).

4.1 Goodness-of-fit

In the upper panel of Figure 1, we plot the sample mean of the actual index-linked zero yields (maturity from 5 years to 15 years, represented by circles) between 1985 and 2011. We can see again that the long end of the real yield curve slopes slightly upwards, with the 5-year rate at around 3.08% and the 15-year rate at around 3.38%. In the same graph, we also plot the means of the real interest rates, represented by the solid line, from the estimated term structure model. The model provides a very good fit of the average yield curve with small pricing errors. For example, for the 7-year, 10-year and 12-year yields, the root mean square pricing errors are 0.0643%, 0.0580% and 0.0379% per annum respectively.¹² The lower panel of Figure 1 plots the sample standard deviations of the actual index-linked zero yields as well as the standard deviations of the real rates from the estimated model. We can see that the model also provides a good fit to the cross-section of the second moments of the real interest rates.

Most arbitrage-free term structure models with macroeconomic fundamentals assume Gaussian distributions with constant volatility. We find that allowing for stochastic volatility of consumption growth greatly improves the goodness-of-fit of the term structure model. Figure 2 plots the sample mean and sample standard deviations of the actual index-linked zero yields together with the mean yield curve and standard deviations of long-term real rates from an estimated term structure model. The terms structure model is same as the one in Section 3 except that it assumes homoskedasticity in consumption growth. Compared to the model with stochastic volatilities

¹²Root mean square pricing errors are computed as $\sqrt{\sum_{t=1}^T (R_{n,t} - \hat{R}_{n,t})^2 / T}$, where $R_{n,t}$ is the observed n -year index-link zero yield at time t , $\hat{R}_{n,t}$ is the fitted n -year real interest rate from the term structure model.

in consumption growth, we can see that the constant-volatility model does a poor job in matching the average yield curve and especially the second moments of the interest rates. The model substantially underestimates the volatility of long-term real interest rates. A formal likelihood ratio test also strongly favors the model with stochastic volatility.

The estimated term structure model in Section 3 allows us to extract time-varying levels of consumption growth volatility as well as the expected consumption growth from inflation-indexed zero yields. Figure 3 plots the estimated state variable \hat{z}_t for the sample period of Q1.1985 - Q4.2011. z_t drives the conditional variance of the consumption growth rate. Large variations in \hat{z}_t indicate that the conditional variance of Δc_{t+1} changes significantly over time. The correlation coefficient between consumption growth, Δc_t , and its volatility, \hat{z}_t , is around 0.10, suggesting that high volatility states tend to coincide with those where consumption is high. The time-varying growth volatility is also very persistent. The autocorrelation coefficient for \hat{z}_t is about 0.80.

In Figure 4, we plot the expected consumption growth estimated from our model, $\hat{\mu}_t$, together with the quarterly consumption growth rate, Δc_t , for the period of Q1.1985 - Q4.2011. We have subtracted mean growth rate from Δc_t in the graph. Mean consumption growth rate during this period is about 1.87% per year. Compared with Δc_t , the expected consumption growth is smaller but much more persistent. While Δc_t has an autocorrelation coefficient of 0.16, $\hat{\mu}_t$ exhibits strong positive autocorrelation with a coefficient of 0.91. $\hat{\mu}_t$ has a standard deviation of 2.2% per annum compared to 3.2% for consumption growth. To see how well $\hat{\mu}_t$ captures variations in the expected consumption growth, we compute a 2-year moving average of consumption growth as $\overline{\Delta c}_t = \sum_{i=-3}^4 \Delta c_{t+i}$. The correlation between $\hat{\mu}_t$ and $\overline{\Delta c}_t$ is 7.9% for the whole sample and 32% if we exclude the 2008-2009 period.

Arbitrage-free term structure models with latent factors usually use the descriptive properties of the yield curve to characterize the state variables,

such as the level, slope and curvature factors. However, in the current paper, the state variables, $X_t = (z_t, \mu_t)$, have clear economic interpretations in terms of consumption growth. To see how are these consumption-based factors related to yield-curve factors, we regress, respectively, $\hat{\mu}_t$ and \hat{z}_t on the 5-year rate and the spread between the 15-year rate and the 5-year rate. The results are included in Table 5. We can clearly see that the growth volatility, \hat{z}_t , is closely related to the yield spread, and a higher volatility will tend to be associated with a flatten or even invert the yield curve. On the other hand, the expected consumption growth, $\hat{\mu}_t$, is closely related to the level of interest rates. Consistent with economic theories, the coefficient on $Y_{5,t}$ is positive and significant. Moreover, since the yield spread contains information about future interest rates, a positive and significant coefficient on $Y_{15,t} - Y_{5,t}$ implies that $\hat{\mu}_t$ is also related to the level of future interest rate.

4.2 Bond risk premiums

Our model allows us to easily decompose bond risk premiums into different components. Consider an $(n + 1)$ -period bond at time t . Its risk premium, or the expected excess rate of return between t and $t + 1$ can be obtained as

$$rp_{n,t} = (E_t^P - E_t^Q) \ln P_{n,t+1} \quad (25)$$

where $P_{n,t+1}$ is the bond price at $t + 1$, E_t^P represents the conditional expectation under the physical probability measure, \mathbb{P} , and E_t^Q represents the conditional expectation under the risk-neutral probability measure, \mathbb{Q} , defined in (9) above.

Using the model in Section 3, we can easily get

$$rp_{n,t} = -B_n \times [-\phi^* + (\phi - \phi_1^*)\mu_t] - C_n \times [(\psi - \psi^*)\delta + (\rho - \rho^*)z_t] \quad (26)$$

The first part on the right-hand side of (26) represents the part of the

bond risk premium due to time-varying expected consumption growth or the long-run consumption risk, μ_t , and the second term represents the risk premium attributable to time-varying growth volatility, z_t . These risk premiums can also be expressed as the negative of the conditional co-variance between the log bond price, $\ln P_{n,t+1}$ and the log stochastic discount factor (or intertemporal marginal rate of substitution), $\ln M_{t,t+1}$ defined in (7), under, respectively, shocks to the expected growth μ_{t+1} and shocks to the growth volatility z_{t+1} , or¹³

$$rp_{n,t} \approx -\text{Cov}_t^\mu(\log P_{n,t+1}, \log M_{t,t+1}) - \text{Cov}_t^z(\log P_{n,t+1}, \log M_{t,t+1}) \quad (27)$$

When the covariance is negative (positive), the risk premium is positive (negative). This is because a negative covariance implies lower asset returns in states where the marginal utility is high, making it very risky to hold that asset. On the other hand, a positive covariance implies higher asset returns in states where the marginal utility is high, making the asset a hedge against the risk factor.

These risk premiums for the 10-year real bond based on the estimated the term structure model are plotted in Figure 5, and their summary statistics are reported in Table 6. We find that both the long-run consumption risk and the volatility risk are priced in the long-term index-linked bond yields and exhibit strong time variations. The long-run consumption risk is in fact the primary source of the bond risk premium and dominates the volatility risk. The average bond risk premium (annualized) between 1985 and 2011 is around 0.925%, among which 1.275% is the long-run risk premium and -0.350% is the volatility risk premium. The standard deviations of the total bond risk premium, long-run risk premium and volatility risk premium are 0.267%, 0.197% and 0.092% respectively. Since, by construction, the total risk premium is the sum of the long-run risk premium and the volatility risk premium, the OLS regression coefficient of the long-run risk premium (or the volatility risk premium) on the total risk premium provide a good

¹³It is an approximation result because z_t is not Gaussian.

“variance decomposition” of the bond risk premium. As Table 6 shows, the long-run risk accounts more than 70% of the variations of the bond risk premium.

It is interesting to note that the risk premium for the growth volatility is negative throughout the sample period, indicating that long-term real bonds provide an effective hedge against the volatility risk in consumption growth. Moreover, since the estimated market price of the volatility risk is positive, $\hat{\lambda}_z > 0$, it then follows that $\rho - \rho^* > 0$ (see Equation (15)), and hence the risk premium in (26) decreases as the volatility increases (note that $C_n > 0$). When growth uncertainty increases, investors are more willing to hold long-term real bonds which promise constant future consumption. This result is consistent with the finding in Bansal and Shaliastovich (2013).

Although both the long-run risk premium and the volatility risk premium are time-varying as Figure 5 shows, the sources of their time-variations are completely different. The risk premium in (27) can be alternatively written as

$$rp_{n,t} = -B_n \sigma_\mu^2 \lambda_{\mu,t} - C_n \sigma_{z,t}^2 \lambda_z \quad (28)$$

In the equation above, σ_μ^2 is the conditional variance of μ_{t+1} which is constant in our model, $\sigma_{z,t}^2$ is the conditional variance of z_{t+1} which is equal to $\psi^2 \delta + 2\psi \rho z_t$ as in (5), λ_z is the market price of the volatility risk and is constant, $\lambda_{\mu,t}$ is the market price of long-run consumption risk which is given in (17).

The volatility risk premium varies over time because the risk, i.e. the variance of the growth volatility, $\sigma_{z,t}^2$, changes over time while the market price of risk, λ_z , stays constant. In contrast, the long-run risk premium changes over time because the market price of risk, $\lambda_{\mu,t}$ changes over time while the risk, i.e. the variance of the expected consumption growth σ_μ^2 , remains constant.¹⁴ Our estimation shows that the time-varying market price of risk, not the time-varying risk, is the primary source of time-variations in

¹⁴Even if we allow z_t to drive the variance of μ_t , we still get the same decomposition where one part of $rp_{n,t}$ is driven by $\lambda_{\mu,t}$ and the other part is driven by z_t .

the bond risk premium.

Using the estimated parameters we can easily obtain from (17),

$$\sigma_{\mu}^2 \lambda_{\mu,t} = -0.0002 - 0.0061 \mu_t \quad (29)$$

where μ_t is the expected consumption growth. The negative coefficient on μ_t implies a counter-cyclical market price of risk. Note that the risk premium, or the expected excess rate of return, on the long-term real bond is procyclical because of $-B_n$ term in (28). Investors require a higher (lower) excess rate of return on default-free long-term real bonds in good (bad) times. In contrast, the standard long-run risk models assume that the market price of risk is constant, and that risk premiums are time-varying only because the growth risk, as measured by the volatility of (expected) consumption growth, changes over time. One limitation of this restriction is that risk premiums will be perfectly correlated with the consumption growth volatility, and therefore the model may not be able to fully capture the dynamic properties of the real yield curve. For example, one of the implications of the standard long-run risk model is that the real yield curve slopes downwards as pointed by Beeler and Campbell (2012) among others. However, the data show that the yield curve slope has changed significantly over the sample period. The average yield curve has a positive slope for the whole sample period (1985 - 2011) as shown in Table 1. And if we plot the 5-year moving average of the yield spread between the 15-year bond and the 5-year bond (see Figure 6), we can see that the yield curve slope is positive approximately before 1998 and after 2005, and the slope is negative in between 1998 and 2005. With a flexible specification of the market price of risk, our model is able to match the movements in the yield spread and generate a positive slope for the yield curve.

5 Conclusions

Latent-factor term structure models based on the no-arbitrage condition have been the most popular framework for studying the joint dynamics of interest rates of different maturities. These models have rich specifications of time-varying risk premiums and are able to account for many salient features of bond yields. Compared to other economically grounded asset pricing models, the empirical success of these dynamic factor models mainly is due to that they only impose the no-arbitrage condition on the cross-section of bond yields while relaxing other general equilibrium restrictions. The cost of the added econometric flexibilities, however, is that the factors or state variables in these models usually lack clear economic interpretations. They only summarize the statistical properties of the yield curve.

We attempt to bridge this gap by estimating a consumption-based, no-arbitrage model of the term structure of real interest rates. The empirical exercise conducted in this paper is based on equilibrium long-run risk models. On one hand, we retain the same econometric flexibilities of the latent-factor models and are able to obtain a tractable solution of the term structure of real interest rates with time-varying risk premiums and stochastic volatilities. On the other hand, the state variables in our model are linked directly to the expected consumption growth as well as time-varying levels of the growth volatility.

The model allows us to examine *empirically* the role of consumption risk in determining long-term real interest rates. We extract a small but persistent long-run component in consumption growth as well as time-varying levels of growth volatility from consumption growth rate *and* long-term, index-linked bond yields. Consistent with the calibration results of equilibrium long-run risk models, we find that both risks are priced in the bond market. The long-run consumption risk, in fact, dominates the volatility risk and drives most of the variations in the risk premiums of long-term real bonds. The risk premium for consumption volatility is negative, suggesting that long-term real bonds provide an effective hedge against the volatility

risk in consumption growth. In contrast to the standard long-run risk model, however, we find that the stochastic growth volatility alone is not sufficient to account for the time variations in bond risk premiums. Movements of the risk premiums seem to be primarily driven by time-varying market prices of risk.

In this paper, the more general econometric specifications relative to the standard long-run risk model, however, come at the expense of the tight connection between the market prices of risk and the deep structural parameters that characterize investor's preferences. A nature extension of the current paper is to derive time-varying market prices of risk from a general equilibrium model by introducing exogenous preference shocks as in Schorfheide et al. (2013). Such a structural model may help us further understand the economic forces underlying the dynamic behavior of the real yield curve and its relation to various consumption risks. Our empirical exercises is an intermediate step toward achieving this goal. Another extension is to include other long-lived assets such as stocks and nominal bonds in the study along the line of Lettau and Wachter (2011) and Lustig et al. (2013) among other. The expanded asset space can better capture all important sources of aggregate risk that affect investor's stochastic discount factor. These extensions are left for future research.

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A The equilibrium real yield curve under recursive utility function

Consider an economy with complete markets and a representative agent who maximizes a Epstein and Zin (1989) type recursive preferences,

$$U_t = \left[(1 - \beta) C_t^{\frac{1-\gamma}{\theta}} + \beta \left(E_t V_{t+1}^{1-\gamma} \right)^{\frac{1}{\theta}} \right]^{\frac{\theta}{1-\gamma}} \quad (30)$$

subject to her inter-temporal budget constraint,

$$W_{t+1} = R_{c,t+1}(W_t - C_t) \quad (31)$$

where C_t is consumption at time t , W_t is the wealth of the agent at time t , $R_{c,t+1}$ is the gross return on the wealth portfolio. In the utility function, $0 < \beta < 1$, γ is the coefficient of risk aversion, $\theta = \frac{1-\gamma}{1-1/\psi}$ and ψ is the elasticity of intertemporal substitution.

Consumption is assumed to have the following dynamics,

$$\Delta c_{t+1} = \mu_0 + \mu_t + \sqrt{z_t} \varepsilon_{t+1} \quad (32)$$

$$\mu_{t+1} = \phi \mu_t + \sqrt{\sigma_{\mu,0} + \sigma_{\mu,1} z_t} \nu_{t+1} \quad (33)$$

$$z_{t+1} = \bar{z} + \rho(z_t - \bar{z}) + \sigma_z \omega_{t+1} \quad (34)$$

where Δc_{t+1} is the log consumption growth rate and μ_t is a small but persistent component of expected consumption growth. All shocks are i.i.d normal and are orthogonal to each other. As in the long-run risk model of Bansal and Yaron (2004), both Δc_{t+1} and μ_{t+1} have stochastic volatilities that are driven by a common Gaussian state variable z_t .

The Euler equation for this economy is given by:

$$E_t \left[e^{m_{t+1} + r_{j,t+1}} \right] = 1 \quad (35)$$

where $r_{j,t+1}$ is the log of the gross return on asset j , and m_{t+1} is the log of

the intertemporal marginal rate of substitution (or the stochastic discount factor) given by:

$$m_{t+1} = \theta \log \beta - \frac{\theta}{\psi} \Delta c_{t+1} - (1 - \theta) r_{c,t+1} \quad (36)$$

The the log return on the consumption claim, $r_{c,t+1} = \ln R_{c,t+1}$, can be approximated by, as in Campbell and Shiller (1988),

$$r_{c,t+1} = k_0 + k_1 x_{t+1} - x_t + \Delta c_{t+1} \quad (37)$$

where x_t is the log price/consumption ratio, k_0 and k_1 are log linearization constants.

Applying the Euler equation to the consumption claim, we can solve for the price/consumption ratio as

$$x_t = A_0 + A_1 \mu_t + A_2 z_t \quad (38)$$

where A_i , $i = 0, 1, 2$, are functions of the model parameters.

Having solved x_t , the intertemporal marginal rate of substitution now becomes:

$$\begin{aligned} m_{t+1} = & \theta \log \beta - (1 - \theta)(k_0 - (1 - k_1)A_0) + (1 - \theta)A_1 \mu_t + (1 - \theta)A_2 z_t \\ & - \gamma \Delta c_{t+1} - (1 - \theta)k_1 A_1 \mu_{t+1} - (1 - \theta)k_1 A_2 z_{t+1} \end{aligned} \quad (39)$$

We can easily see that,

$$\begin{aligned} m_{t+1} - E_t m_{t+1} = & -\gamma(\Delta c_{t+1} - E_t \Delta c_{t+1}) - (1 - \theta)k_1 A_1 (\mu_{t+1} - E_t \mu_{t+1}) \\ & - (1 - \theta)k_1 A_2 (z_{t+1} - E_t z_{t+1}) \end{aligned} \quad (40)$$

Therefore, with the recursive utility function, the market prices of short-run consumption risk, long-run consumption risk and volatility risk are given

by, respectively,

$$\lambda_c = \gamma \quad (41)$$

$$\lambda_\mu = (1 - \theta)k_1 A_1 \quad (42)$$

$$\lambda_z = (1 - \theta)k_1 A_2 \quad (43)$$

All market prices of risk are constant. Risk premiums are time-varying because the volatilities of (expected) consumption growth, Δc_{t+1} and μ_{t+1} , are changing over time and are driven z_t .

To solve for the term structure of real interest rates, we first note that the one-period real interest rate can be obtained from the Euler equation:

$$e^{-r_{1,t}} = E(e^{m_{t+1}}) \quad (44)$$

and

$$r_{1,t} = \alpha_0 + \alpha_1 \mu_t + \alpha_2 z_t \quad (45)$$

for some constants α_i , $i = 0, 1, 2$.

The interest rate on an n -period bond can be obtained recursively,

$$e^{-nr_{n,t}} = E_t \left(e^{m_{t+1} - (n-1)r_{n-1,t+1}} \right) \quad (46)$$

or

$$e^{-nr_{n,t}} = e^{-r_{1,t}} E_t^Q \left(e^{-(n-1)r_{n-1,t+1}} \right) \quad (47)$$

where E_t^Q refers to the conditional expectation taken with respect to the risk-neutral probability measure determined by m_{t+1} . It can be easily shown that

$$r_{n,t} = \alpha_{n,0} + \alpha_{n,1} \mu_t + \alpha_{n,2} z_t \quad (48)$$

for some constants $\alpha_{n,0}$, $\alpha_{n,1}$ and $\alpha_{n,2}$ that satisfy a system of difference equations in n .

Table 1 Summary Statistics: consumption and real yield curve

	Δc_t	$R_{5,t}$	$R_{7,t}$	$R_{10,t}$	$R_{12,t}$	$R_{15,t}$
Mean	1.8717	3.0845	3.1879	3.2882	3.3320	3.3761
Std. Dev.	3.2092	0.6560	0.6156	0.5927	0.5916	0.6010
Auto Corr	0.1589	0.7799	0.8096	0.8458	0.8692	0.8985

Δc_t is annualized quarterly growth rate of real per-capita consumption on non-durable goods and services. $R_{i,t}$ ($i = 5, 7, 10, 12, 15$) is i -year real interest rate extracted from prices of inflation-indexed government bonds. Interest rates and consumption growth rates are all in percentage points. The sample period is Q1.1985 - Q4.2011.

Table 2 Cross Correlations: consumption and real interest rates

	Δc_{t-8}	Δ_{t-4}	Δc_{t-1}	Δc_t	Δc_{t+1}	Δc_{t+4}	Δc_{t+8}
$R_{5,t}$	0.1640	0.2285	0.1380	0.1992	0.2186	0.2734	0.3405
$R_{10,t}$	0.0974	0.1721	0.0961	0.1727	0.1716	0.2023	0.2464
$R_{15,t}$	0.0425	0.0861	0.0311	0.0917	0.1009	0.1097	0.1452

Δc_t is annualized quarterly growth rate of real per-capita consumption on non-durable goods and services. $R_{i,t}$ ($i = 5, 10, 15$) is i -year real interest rate extracted from prices of inflation-indexed government bonds. The sample period is Q1.1985 - Q4.2011.

Table 3 Forecasting Consumption Growth

	$R_{5,t}$	Δc_t	Δc_{t-1}	R^2
Δc_{t+1}	0.9543* (0.3800)	0.1202 (0.1126)		0.0617
$\overline{\Delta c_{t \rightarrow t+4}}$	0.9062* (0.3132)	0.1269 (0.0690)	0.1125* (0.0561)	0.2008
$\overline{\Delta c_{t \rightarrow t+8}}$	1.0962* (0.2362)	0.1015* (0.0417)	0.1248* (0.0341)	0.3688

This table reports the OLS regression of consumption growth on lagged interest rates. Δc_t is annualized quarterly growth rate of real per-capita consumption on non-durable goods and services. $\overline{\Delta c_{t \rightarrow t+i}}$ is the average consumption growth rate between quarter t and quarter $t+i$, $R_{5,t}$ is the 5-year real interest rates extracted from prices of inflation-indexed government bonds. Numbers in parentheses are heteroskedasticity and autocorrelation consistent standard errors. Regression coefficients with * indicate statistically significant at 5% level. The sample period is Q1.1985 - Q1.2011.

Table 4 Parameter Estimates

δ	1.4946 (0.0464)
ρ	0.9588 (0.0041)
ψ	0.00046 (0.00004)
μ_0	0.0165 (0.0015)
ϕ	0.9955 (0.0005)
σ_μ	0.0024 (0.0003)
A_1	0.0049 (0.0005)
B_1	0.3544 (0.1197)
λ_z	4.7501 (0.4936)
ϕ_0^*	0.0002 (0.0016)
ϕ_1^*	1.0016 (0.0196)
σ_{15}	0.0128 (0.0038)
Likelihood Function	1232.53

This table reports the maximum likelihood estimates of the parameters of the term structure models in Section 3. Standard errors are included in the parentheses.

Table 5 Yield Curve Factors

	$Y_{5,t}$	$Y_{15,t} - Y_{5,t}$	R^2
$\hat{\mu}_t$	2.8106* (0.1080)	4.6885* (0.3430)	0.95
\hat{z}_t	-0.0279 (0.0595)	-2.6116* (0.1903)	0.95

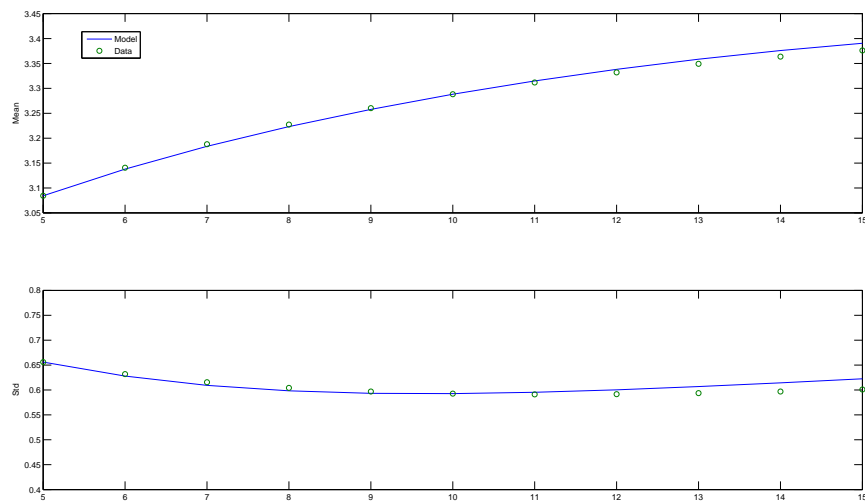
This table reports the OLS regression of the expected consumption growth, $\hat{\mu}_t$, and the growth volatility, \hat{z}_t , on the 5-year rate, $Y_{5,t}$, and the spread between the 15-year rate and the 5-year rate, $Y_{15,t} - Y_{5,t}$. Heteroskedasticity and autocorrelation consistent standard errors are included in the parentheses. $\hat{\mu}_t$ and \hat{z}_t are based on maximum likelihood estimates of the term structure models. Regression coefficients with * indicate statistically significant at 5% level.

Table 6 Time-varying Bond Risk Premiums

	Total Risk	Long-run Risk	Volatility Risk
Mean	0.9248	1.2749	-0.3501
Standard Deviation	0.2670	0.1966	0.0916
Variance Ratio	1	0.7123	0.2877

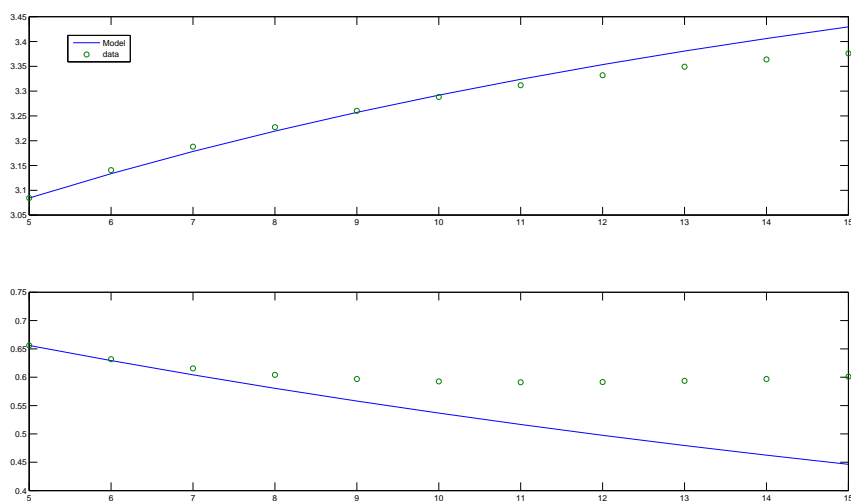
Total Risk is the annualized expected excess holding-period return for a 10-year real bond defined in (25). Long-run Risk represents the part of risk premium due to time-varying expected consumption growth. Volatility Risk represents the part of the risk premium due to time-varying consumption growth volatility. Total risk premium is the sum of the long-run consumption risk premium and the volatility risk premium. Variance Ratio is the the ratio of co-variance between total risk premium and the long-run consumption risk premium (or the volatility risk premium) over the variance of total risk premium.

Figure 1: The Estimated Long-end of the Real Yield Curve



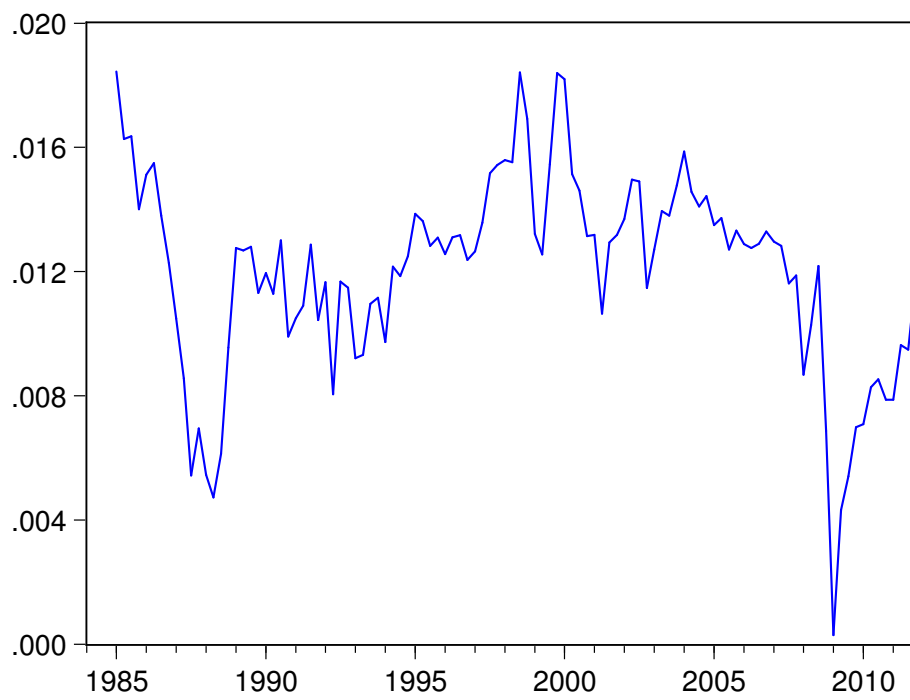
Upper panel of the figure plots the average of yield curve. The lower panel of the figure plots the standard deviations of real rates. The maturities are in years. The solid lines are from the model in Section 3. Circles represent data. The sample period is from 1985 to 2011.

Figure 2: The Estimated Long-end of the Real Yield Curve: constant volatility



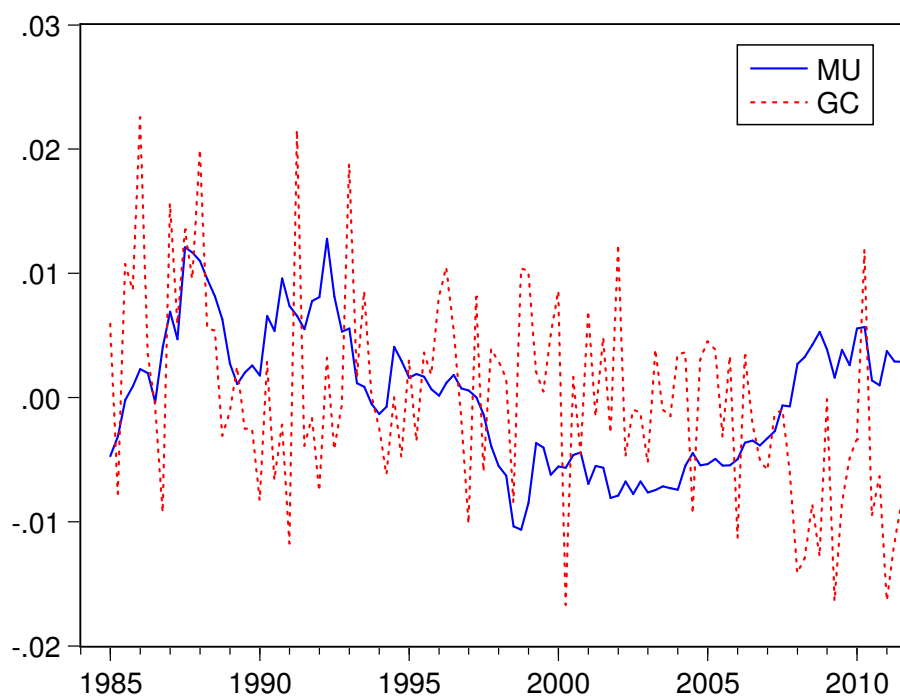
Upper panel of the figure plots the average of yield curve. The lower panel of the figure plots the standard deviations of real rates. The maturities are in years. The solid lines are from the same model as in Section 3 except that volatility is constant. Circles represent data. The sample period is from 1985 to 2011.

Figure 3: Time-varying Consumption Growth Volatility



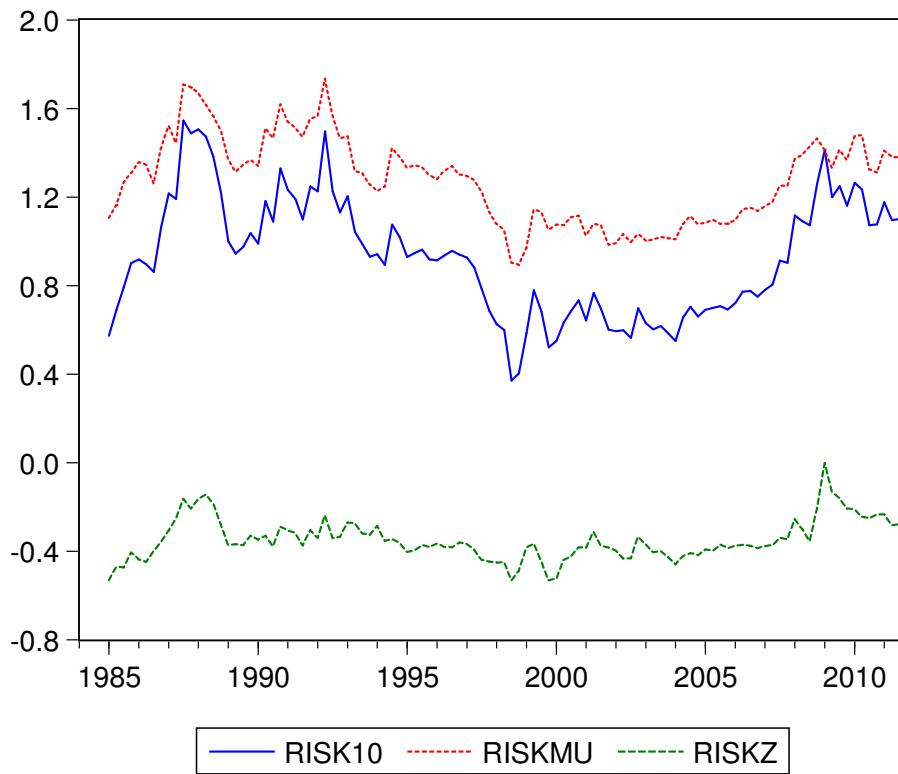
The figure plots the estimated state variable z_t for the period Q1.1985 - Q4.2011. z_t drives the stochastic volatility of consumption growth.

Figure 4: Expected Consumption Growth



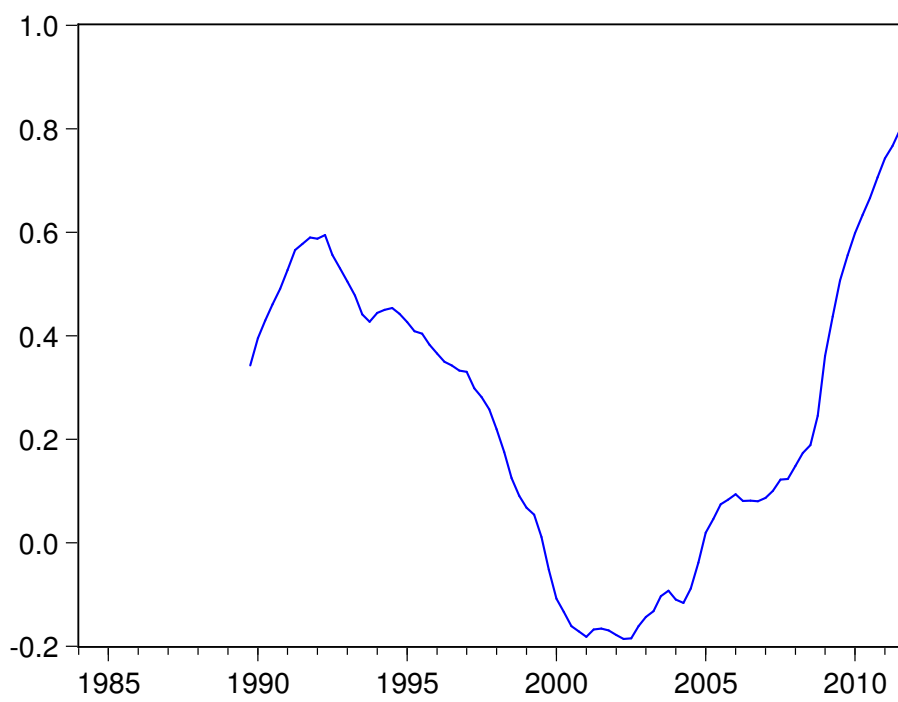
The figure plots (de-meanned) consumption growth rate (the dotted line, denoted as gc) together with the expected consumption growth (the solid line, denoted as mu) estimated from the model. The sample period is Q1.1985 - Q4.2011.

Figure 5: Time-varying Risk Premiums



The figure plots the risk premiums of a 10-year real bond (the expected excess holding-period-return). RISK10 is the total risk premium, RISKMU represents to the part of the risk premium due to time-varying expected consumption growth or long-run consumption risk, RISKZ represents to the part of the risk premium due to time-varying consumption growth volatility.

Figure 6: Real Yield Curve Slope



The figure plots the 20-quarter moving average of the yield spread between the 15-year real bond and the 5-year real bond.